Contents

L1 Basics

1.1 A - Historical Perspective

Position:

1543 Nicolaus Copernicus

1588 Tycho Brahe

1609 Johannes Kepler

Movement: Newton:

- Law of universal gravitation
- Laws of motion
- Calculus

1.2 B - Basic Concepts of Motion and Gravity

1.2.1 Newton's Laws of Motion

1. Every particle continues in its state of rest or uniform motion in a straight line relative to an inertial reference frame, unless it is compelled to change that state by forces acting upon
2. The time rate of change of linear momentum of a particle relative to an inertial reference frame is proportional to the resultant of all forces acting upon that particle and is colinear with and in the direction of the resultant force

$$\vec{F}_{res} = \sum_i \vec{F}_i = \sum \frac{d}{dt}\left(m_i \vec{V}_i\right) \tag{1.1}$$

3 If two particles exert forces on each other, these forces are equal in magnitude and opposite in direction

1.2.2 Inertial Reference Frames

Suppose we want to describe the motion of a particle, p. Ideally we want to describe it with respect to an inertial reference frame: a frame with respect to which a particle remains at rest or moves in a straight line with a constant speed if no external force acts upon the particle. This is described by a Galilean transformation. Time is universal, e.g. the only difference might be a constant offset.

$$\vec{r}' = \vec{r}. - \vec{W}(t - t_0) \tag{1.2}$$

$$t' = t + T \tag{1.3}$$

We solve the problem of inconvenient modeling of the velocity (e.g. due to a rotating frame) by introducing:

- Apparent forces
  - Centrifugal force
  - Coriolis force
- External forces
  - Rocket thrust

1.2.3 Newton's Second Law

Applying the Galilean transformation:

$$\vec{F} = \frac{d}{dt}(m\vec{V}) \tag{1.4}$$

$$\vec{F} = \frac{d}{dt}(m\vec{V} - \vec{W}\frac{dm}{dt}) \tag{1.5}$$

Assumption: the mass is constant. Then:

$$\vec{F} = m\frac{d\vec{V}}{dt} \tag{1.6}$$

Or:

$$\vec{F} = M\frac{d\vec{V}_{cm}}{dt} \tag{1.7}$$

Eventually we will have many bodies:

$$\sum \vec{F} = m\vec{a} \tag{1.8}$$

From that, including thrust:

$$\vec{F} - \dot{m}\vec{V}_j = M\frac{d\vec{V}_{cm}}{dt} \tag{1.9}$$

Note that V_j is relative to the spacecraft.

We assume deterministic motion. In the three-body problem, chaos theory was found.

1.2.4 Newton's Law of Gravitation

$$\vec{F} = G\frac{m_1 m_2}{r^2}\hat{r} \tag{1.10}$$

Or:

$$\vec{F} = G\frac{m_1 m_2}{r^3}\vec{r} \tag{1.11}$$

G is not easy to measure. Because of this, we usually use the gravitational parameter $\mu = Gm_{\text{Central Body}}$, which can be measured quite accurately.

1.2.5 Gravitational Fields

Denoting the potential $-G\frac{m_1}{r} = U$, we get the gravitational field:

$$\vec{g}_2 = -\vec{\nabla}_2 U_2 \tag{1.12}$$

From this, we get a negative potential at any finite distance.

Note that for the gravitational field, there is near no difference between a thin shell and a point mass. Bodies can be modeled as a sum of different spherical shells:

$$U_P = -\frac{G}{l}\sum_i M_i = -\frac{GM_T}{l} \tag{1.13}$$

$$F_P = -\frac{GM_T m_P}{l^2} \tag{1.14}$$

From this, we can assume that bodies are point masses.

1.3 C - The N-Body Problem

The first step is to use Newton's second law. Then:

$$m_i\frac{d^2\vec{r}_i}{dt^2} = \sum_{j\neq i} G\frac{m_i m_j}{r_{ij}^3}\vec{r}_{ij} \tag{1.15}$$

This gives n equations for n bodies. To solve, we can numerically integrate the equations.

1.3.1 Integrals of Motion

Summing the equations of motion over all n bodies, we can find a constant total linear momentum of the system, and the center of mass of the system remains at rest or experiences uniform rectilinear motion.

We end up with:

$$x_{cm} = a_1 t + b_1 \tag{1.16}$$
$$y_{cm} = a_2 t + b_2 \tag{1.17}$$
$$z_{cm} = a_3 t + b_3 \tag{1.18}$$

Taking the vector product over all n bodies of $\vec{r}_i$ and summing over all i, we can find that the total angular momentum of the system is also constant.

We get:

$$H_x = \sum_i m_i \left(y_i \dot{z}_i - z_i \dot{y}_i\right) = c_1 \tag{1.19}$$
$$H_y = \sum_i m_i \left(z_i \dot{x}_i - x_i \dot{z}_i\right) = c_2 \tag{1.20}$$
$$H_z = \sum_i m_i \left(x_i \dot{y}_i - y_i \dot{x}_i\right) = c_3 \tag{1.21}$$

This defines the invariable plane of Laplace. It is only invariable if there are no external forces and no internal losses in the n-body system. These effects are real in the Solar system, but very small.

Next, summing the scalar product of $\frac{d\vec{r}_i}{dt}$ and Equation 1.15 over all i, we can find that the total mechanical energy of the system is constant.

We get:

$$\sum_i \frac{1}{2} m_i V_i^2 - \frac{1}{2} G \sum_i \sum_{j \neq i} \frac{m_i m_j}{r_{ij}} = C \tag{1.22}$$

We thus have 10 integrals of the motion.

Rewriting Equation 1.15 to the barycentric form:

$$\frac{d^2 \vec{r}_i}{dt^2} = -G \frac{M}{r_i^3} \vec{r}_i + G \sum_{j \neq i} m_j \left(\frac{1}{r_{ij}^3} - \frac{1}{r_i^3}\right) \vec{r}_{ij} \tag{1.23}$$

The first term on the right-hand side, is the two-body form. The second term is the deviations from the motion because of all other particles.

1.3.2 Polar Moment of Inertia, Evolution, Total Collision

If the total energy of the system is larger than or equal to 0,the system is always unstable. If it is negative, it can be stable or unstable.

1.3.3 Pseudo-Inertial Reference Frames

We are always forced to use pseudo-inertial reference frames. The selection of the correct one depends on the motions with respect to the timescale of the problem, the masses and distances of the bodies, etc.

1.3.4 Angular Momentum of the Two-Body Problem

Placing a reference frame at the barycenter of the system, we can find an expression relating the two $\vec{r}_i$. From this, we can find that the shape of the trajectories are the same, and that they show some sort of symmetry.

$$m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0 \tag{1.24}$$

Looking at the angluar momentum:

$$\vec{H} = m_1\vec{r}_1 \times \frac{d\vec{r}_1}{dt} + m_2\vec{r}_2 \times \frac{d\vec{r}_2}{dt} = \vec{c} \tag{1.25}$$

Some observations:

- $\vec{H}_1$ and $\vec{H}_2$ have the same direction
- Both bodies move in the same fixed plane about the barycenter, and about each other in that plane
- The magnitude of $\vec{H}_i$ for each body is constant
- Bodies will always be positioned diametrically opposite each other
- Orbits of the bodies about the barycenter is the same shape

1.4 E - N-Body Relative Motion

We introduce a new reference frame to develop equations of motion with respect to a particle k. Then: change of notation

$$\frac{d^2\vec{r}_i}{dt^2} = -G\frac{m_i + m_k}{r_i^3}\vec{r}_i + G\sum_{j\neq i,k} m_j \left(\frac{vecr_j - \vec{r}_i}{r_{ij}^3} - \frac{\vec{r}_j}{r_j^3}\right) \tag{1.26}$$

This new reference frame is not an inertial reference frame. We can identify three terms:

1. The acceleration of i due to k (two-body term)
2. The acceleration of i due to j (principal part of perturbation)
3. The acceleration of j due to k (indirect part of perturbation)

The magnitude of the perturbing acceleration is:

$$a_m = G\frac{m_k}{r_i^2} \tag{1.27}$$

And the disturbing acceleration (body j):

$$a_d = Gm_d\sqrt{\left(\frac{\vec{r}_{id}}{r_{id}^3} - \frac{\vec{r}_d}{r_d^3}\right)\cdot\left(\frac{\vec{r}_{id}}{r_{id}^3} - \frac{\vec{r}_d}{r_d^3}\right)} \tag{1.28}$$

Which gives:

$$a_d = G\frac{m_d}{r_d^2}\sqrt{q + \frac{1}{(1-2\gamma\cos\alpha+\gamma^2)^2} - \frac{2(1-\gamma\cos\alpha)}{(1-2\gamma\cos\alpha+\gamma^2)^{3/2}}} \tag{1.29}$$

with

$$\gamma = \frac{r_i}{r_d} \tag{1.30}$$

This equation is maximum for $\alpha = 0°$:

$$a_{d_{max}} = G\frac{m_d}{r_d^2}\left|\left(\frac{1}{1-\gamma}\right)^2 - 1\right| \tag{1.31}$$

1.4.1 Sphere of Influence

The sphere of influence is the volume of space around a body within which the motion of a small object can be best described as a perturbed two-body trajectory around this body.

We can find that:

$$\frac{R_{SOI}}{\rho} \approx \left(\frac{m_1}{m_2}\right)^{2/5} \tag{1.32}$$

If this ratio is more than 1, it is not in the SOI. If less, it is.

L2 Two-Body Problem

2.1 A - Equations of Motion

From the EoM for n bodies, we can find the motion w.r.t. body k:

$$\frac{d^2\vec{r}_{ki}}{dt^2} = -g\frac{m_k + m_i}{r_{ki}^3}\vec{r}_{ki} \tag{2.1}$$

And w.r.t. the barycenter:

$$\frac{d^2\vec{r}_{Bi}}{dt^2} = -g\frac{m_k + m_i}{\left(1 + \frac{m_i}{m_k}\right)^3 r_{Bi}^3}\vec{r}_{Bi} \tag{2.2}$$

If body i has negligible mass:

$$\frac{d^2\vec{r}}{dt^2} = -\frac{\mu}{r^r}\vec{r} \tag{2.3}$$

Note that μ is defined differently for the two cases.

2.1.1 Kepler's Laws of Planetary Motion

1. The orbits of the planets are ellipses with the Sun at one focus
2. The line joining a planet to the Sun sweeps out equal areas in equal times
3. The square of the period of the planet is proportional to the cube of its mean distance from the Sun

2.1.2 Newton's Law of Universal Gravitation

Every point mass exerts a gravitational force on every other point mass.

$$\vec{F}_{12} = -\frac{Gm_1m_2}{r_{12}^2}\hat{r}_{12} \tag{2.4}$$

$$\vec{F}_{21} = -\frac{Gm_2m_1}{r_{21}^2}\hat{r}_{21} \tag{2.5}$$

Consider two masses. We can get:

$$\ddot{\vec{r}}_{12} = \ddot{\vec{r}}_2 - \ddot{\vec{r}}_1 \tag{2.6}$$

Then, we can apply Newton's second law:

$$\sum \vec{F} = m\ddot{\vec{r}} \tag{2.7}$$

Thus, for each of the two masses:

$$\vec{F}_{12} = -\frac{Gm_1m_2}{r_{12}^2}\hat{r}_{12} = m_2\ddot{\vec{r}}_2 \tag{2.8}$$

$$\vec{F}_{21} = -\frac{Gm_2m_1}{r_{21}^2}\hat{r}_{21} = m_1\ddot{\vec{r}}_1 \tag{2.9}$$

Simplifying and substituting gives:

$$\ddot{\vec{r}}_{12} = -G\frac{m_1 + m_2}{r_{12}^2}\hat{r}_{12} \tag{2.10}$$

Assuming $m_2 << m_1$:

$$\ddot{\vec{r}}_{12} = -\frac{\mu}{r_{12}^2}\hat{r}_{12} \tag{2.11}$$

Removing the subscripts can be done, which describes the restricted two-body problem. A solution is the trajectory equation.

Some observations:

1. Second-order differential equation

2. Vector equation
3. Double integration produces six constants of integration

Restricted two-body problem:

1. Only gravitational force of m_1 on m_2 is considered
2. $m_2 << m_1$

2.2 B - Trajectory Equation

Two integrals derive from the derivation:

1. Eccentricity
2. Angular momentum
3. Alternatively: semi-major axis

2.2.1 Eccentricity Vector

Used for regularization and orbit determination.

$$\vec{e} = \frac{1}{\mu}\left[\left(V^2 - \frac{\mu}{r}\right)\vec{r} - \left(\vec{r}\cdot\vec{V}\right)\vec{V}\right] \tag{2.12}$$

After some analytical manipulations:

$$e^2 = 1 - \frac{rV^2}{\mu}\left(2 - \frac{rV^2}{\mu}\right)\cos^2\gamma \tag{2.13}$$

The angle the plane makes with the horizontal is closely related to the eccentricity.

$$r = \frac{h^2/\mu}{1 + e\cos\theta} \tag{2.14}$$

Also:

$$p = \frac{h^2}{\mu} \tag{2.15}$$

The Keplerian elements are:

1. p
2. a
3. e
4. θ
5. r

Looking at the periapsis:

$$r_p = \frac{p}{1+e} \tag{2.16}$$

For the apoapsis:

$$r_a = \frac{p}{1-e} \tag{2.17}$$

Also:

$$2a = r_p + r_a \tag{2.18}$$
$$p = a(1 - e^2) \tag{2.19}$$
$$r = \frac{a(1 - e^2)}{1 + e\cos\theta} \tag{2.20}$$
$$e = \frac{r_a - r_p}{r_a + r_p} \tag{2.21}$$

2.2.2 Velocity

The velocity is tangential to the trajectory at any point. It consists of a radial component and a transverse component:

$$\dot{r} = V\sin\gamma \tag{2.22}$$
$$r\dot{\theta} = V\cos\gamma \tag{2.23}$$

γ is the flight path angle. We also have the specific angular momentum $\vec{h} = \vec{r} \times \vec{V}$. From that:

$$h = rV\cos\gamma \tag{2.24}$$

We can make γ a function of the true anomaly θ. This function is different for different eccentricities.

The velocity hodograph shows the velocity vector traced in space over time. It is a plot of $\dot{r}$ versus $r\dot{\theta}$. They are useful for a qualitative analysis of optimal transfer trajectories, rendezvous trajectories, coordinate transformations for trajectory optimizations, and more.

2.3 C - Elliptical Trajectories

2.3.1 Velocity and the Orbital Period

The Vis-Viva equation is given by:

$$E = -\frac{\mu}{2a} = \frac{V^2}{2} - \frac{\mu}{r} \tag{2.25}$$

The first term is the specific kinetic energy, and the second the specific potential energy. From this the velocity at any radius can be found. We can also make many equations useful for manipulations and derivations.

2.3.2 Circular Orbits

For circular orbits, $e = 0$. Then, from the energy equation, we can try to find V_c:

$$V = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}} \tag{2.26}$$
$$V_c = \sqrt{\frac{\mu}{r}} \tag{2.27}$$

The period is found using (not just circular):

$$T = 2\pi\sqrt{\frac{a^3}{\mu}} \tag{2.28}$$

Periods can be:

1. Sidereal
2. Synodic

3. Anomalistic
4. Draconic

We can see that V_c varies by planet, but the period does not for LXO.

2.3.3 Elliptical Trajectories

Just because a spacecraft has V_c at a point P, does not mean it is in a circular orbit: an elliptical trajectory will range from lower to higher velocities, and thus it will have a point with V_c as well, but it will be in a different direction. However, every orbit through point P with the same velocity V_c has the same orbital period.

From the vis-viva equation, we can find that the minimum velocity is found at maximum distance, i.e. at the apoapsis:

$$V_a^2 = \mu \left(\frac{2}{a(1+e)} - \frac{1}{a} \right) = V_{c_a}(1-e) \tag{2.29}$$

Similarly, velocity is maximum at minimum distance, i.e. at the pericenter:

$$V_p^2 = \frac{\mu}{a} \left(\frac{1+e}{1-e} \right) = V_{c_p}(1+e) \tag{2.30}$$

2.3.4 Mean Motion

We again start with the period:

$$T = 2\pi \sqrt{\frac{a^3}{\mu}} \tag{2.31}$$

Define the mean angular motion of the satellite in orbit as:

$$n = \frac{2\pi}{T} \tag{2.32}$$

Then, from Kepler's third law, we can state that:

$$\frac{a^3}{T^2} = \frac{\mu}{4\pi^2} \tag{2.33}$$

with here $\mu = G(m_k + m_i)$. Substituting that in:

$$\frac{a^3}{T^2} = \frac{Gm_k}{4\pi^2} \left(1 + \frac{m_i}{m_k} \right) \tag{2.34}$$

From this, one approach can be created to weigh some planet.

2.4 D - Parabolic Trajectories

For parabolas, $e = 1$. This means that $a = \infty$, and $E = 0$. Generally, the point of closest approach, r_p is of interest. We can find using some previous equations:

$$r_p = \frac{p}{2} \tag{2.35}$$

For all points along the parabola:

$$\gamma = \frac{\theta}{2} \tag{2.36}$$

We can also find:

$$V = \sqrt{2} V_c = V_{esc} \tag{2.37}$$

We can find that:

$$V_{max} = 2\sqrt{\frac{\mu}{p}} \tag{2.38}$$

Other observations:

1. $V = \sqrt{2} V_c$

2. $V_\perp$ decreases monotonically from r_p to r_∞
3. $V_r = 0$ at $\theta = 0°$ and $\pm 180°$
4. $V_r = \max$ at $\theta = 90°$ and $270°$

The parabola has precisely the minimum velocity necessary to escape the central body: the escape velocity V_{esc}.

The size of the parabola depends on the direction of the velocity vector, the flight path angle:

$$p = \frac{4\mu}{V_{esc}^2} \cos^2 \gamma \tag{2.39}$$

2.5 E - Hyperbolic Trajectories

For hyperbolas, $e > 0$, $a < 0$, and $E > 0$. A key characteristic is that hyperbolic trajectories do not cover all available two-dimensional space.

$$\cos\theta > -\frac{1}{e} \tag{2.40}$$

This is because r and p are always positive.

Observations of the velocity:

1. $V_{\perp,max}$ at r_p
2. $V_{\perp,min}$ at $\cos\theta > -1/e$
3. $V_r = 0$ at $\theta = 0°$ and $\pm 180°$
4. $V_r = \max$ at $\theta = 90°$ and $270°$

The velocity reaches a maximum at the periapsis:

$$V_p^2 = V_{c_p}^2 (e + 1) \tag{2.41}$$

The velocity reaches a minimum at $r = \infty$:

$$V_\infty = -\frac{\mu}{a} \neq 0 \tag{2.42}$$

This is also called the hyperbolic excess velocity. Other handy relations:

$$\frac{V_p}{V_\infty} = \sqrt{\frac{e+1}{e-1}} \tag{2.43}$$

$$\frac{V_p}{V_{c_p}} = \sqrt{e+1} \tag{2.44}$$

$$V^2 = V_{esc}^2 + V_\infty^2 \tag{2.45}$$

L3 Reference Systems

3.1 A - Orbital Elements

From the restricted two-body problem, we know we need 6 constants. To describe the motion in 3D, we need a reference frame. We need to define the orientation with respect to fiducial points. We assume that the origin is at the center of the central body around whose gravitational force the spacecraft is obliged to move. The inclination i is defined as the angle between the $x - y$ plane and the orbital plane.

1. $0° \leq i < 90°$: Direct / Prograde
2. $90° < i \leq 180°$: Indirect / Retrograde

The right ascension of the ascending node Ω is the angle rotated about the z axis of the orbital plane.

The argument of the periapsis ω defines how far the orbital plane is rotated about its angular momentum vector.

3.1.1 Orbital Singularities

If $e = 0$, there is no periapsis, which makes ω undefined. We define u as the argument of latitude, equal to $\omega + \theta$.

If $i = 0°$, no Ω is defined. We then define Π as the longitude of periapsis, equal to $\Omega + \omega$.

If $i = 0°$ and $e = 0$, we define ℓ as the true longitude, equal to $\Omega + \omega + \theta = \Pi + \theta = \Omega + u$.

3.2 B - Common Terms and Reference Frames

We already covered the right ascension of the ascending node, argument of periapsis, inclination of the orbit. We also have the obliquity of the ecliptic.

3.2.1 Terminology

1. North Pole (upwards rotation axis)
2. South Pole (downwards rotation axis)
3. Zenith (surface outwards)
4. Nadir (surface inwards)
5. Meridian (arc from North to South)
6. Parallel (intersection of planets spheroid)
7. Equator
8. Geographic longitude
9. Geocentric latitude
10. Geodetic latitude
11. Ecliptic
12. Equinox line
13. Vernal / autumnal equinoxes
14. Declination
15. Right ascension

3.2.2 Reference Systems

A reference system is a complete specification of how a coordinate system is to be formed. The origin and orientation of fundamental reference axes or planes are defined, and it includes a specification of the fundamental models required to construct the system, and of the epoch (time) at which the system is formed.

A reference frame are coordinates of a set of fixed, fiducial points which serve as the practical realization of the reference system. A reference frame is subsequently used only to indicate the reference by which a coordinate system is defined.

3.2.3 Topocentric Reference Frame

The topocentric reference frame has the observer as the origin. The fundamental plane is the local horizontal plane.

1. $\hat{x}$ positive due North
2. $\hat{z}$ positive Nadir
3. $\hat{y}$ completes RHS

We have the azimuth ψ, elevation H, and the slant range ρ.

3.2.4 Other Key Reference Frames

1. Spacecraft Local Horizontal
2. Geocentric Rotating (ECEF)

Some non-rotating frames:

1. Non-rotating Geocentric Equatorial
2. Non-rotating Heliocentric Ecliptic

3.3 C - Position on the Surface

The longitude is due East from the prime meridian, and the geocentric latitude is due North from the equator.

3.3.1 Great Circle

A great circle of a sphere is the intersection of the sphere and a plane that passes through the center point of the sphere. It is the largest circle that can be drawn on any given sphere.

3.3.2 Longitude and Latitude

The longitude Λ is the angle between the local meridian and the prime meridian.

The geocentric latitude β is the angle is the angle between the local circular plane and the equatorial plane. The geodetic latitude Φ is defined because of oblateness of planets. It is found from the normal line to the local.

The flattening is found by:

$$f = \frac{a_e - a_p}{a_e} \tag{3.1}$$

Similar to eccentricity.

There also is the astronomical latitude which is the line along the true local vertical.

The station error is the difference between the geodetic and the astronomical.

3.3.3 ITRF

Earth has many irregularities. Because of this, the international terrestrial reference frame (ITRF) was defined.

3.3.4 Polar Motion

The movement of the point where the Earth's rotation axis intersects its surface. It is driven by continuous mass redistribution in the atmosphere, oceans, mantle, and core. The amplitude is up to hundreds of milli-arcseconds or a few meters on the Earth's surface.

3.4 Astronomical Reference Frames & Coordinates

To locate a celestial object, we make use of the celestial sphere. The celestial meridian rotates with Earth. The hour circle is the meridian on which a celestial object lies. The difference is the hour angle.

The celestial pole is an extension of the North axis. Then the $\hat{z}$ axis is defined by the vernal equinox, and the $\hat{y}$ axis completes the RHS.

The celestial equator is an extension of the equatorial plane. The intersection of the ecliptic and the celestial equator is the equinox line.

We can define a position using the right ascension and the declination δ with respect to the celestial equator. In the ecliptic, we use the celestial longitude and the celestial latitude.

Precession is the movement of the Earth's axis of rotation, and thus the equator, with respect to the celestial sphere.

Nutation is in-out, precession is rotation.

Luni-solar precession is a smooth long-period motion of the Earth's mean celestial north pole around the ecliptic north pole. It causes the vernal equinox to move along the ecliptic.

Luni-solar nutation is a relatively short-period motion of the true celestial pole around the mean celestial pole. It causes the obliquity of the ecliptic to change, and the vernal equinox to oscillate about the ecliptic.

Because of these, the fully specified reference system is defined by the mean of [date], usually the year 2000.

3.5 G - Position and Time

The trajectory equation relates r to θ. Geometry, position, velocity, energy, and angular momentum have been discussed, but the relation to time has been absent so far. It starts with Kepler's second and third laws.

A direct relation between θ and t is difficult to obtain. Instead, we use an auxiliary circle, where the semi-major axis of the ellipse equals the radius of the circle. We define E as the eccentric anomaly, which is the anomaly of this new circle. From this:

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} \tag{3.2}$$

The mean anomaly M is defined as:

$$M = E - e\sin E \tag{3.3}$$

The mean motion n is the mean angular motion of a body:

$$n(t-\tau) = E - e\sin E \tag{3.4}$$

$$n = \sqrt{\frac{\mu}{a^3}} \tag{3.5}$$

To go from θ to t is straight forward. The other way around is not, and we need to solve it iteratively:

$$E_{i+1} = M + e\sin E_i \tag{3.6}$$

It can also be solved using a graphical method.

For parabolas, we can use Barker's equation:

$$\tan\frac{\theta}{2} + \frac{1}{3}\tan^3\frac{\theta}{2} = 2\sqrt{\frac{\mu}{p^3}}(t-\tau) \tag{3.7}$$

We get an analog of Kepler's third law:

$$\frac{p^3}{T_p^2} = \frac{9}{16}Gm_k = \text{ constant} \tag{3.8}$$

For hyperbolas, we define the hyperbolic anomaly F. We can get:

$$r = a(1 - e\cosh F) \tag{3.9}$$

From this:

$$e\sinh F - F = \bar{M} \tag{3.10}$$

$$\bar{M} = \bar{n}(t-\tau) \tag{3.11}$$

$$\bar{n} = \sqrt{\frac{\mu}{-a^3}} \tag{3.12}$$

We can use Euler's formula to manipulate these.

3.6 F - Lambert's Theorem

Lambert's theorem is used to calculate the time of flight. It is essentially a reformulation of Kepler's equation.

To determine the time of flight between two points in an elliptical orbit, we could compute the eccentric anomalies at both points and use Kepler's equation twice.

A more elegent solution is by seeing that the time needed to traverse an elliptical arc only depends on a, c, and $r_1 + r_2$. This results in Lambert's equation for elliptical motion. Some intermediate results:

$$a_{min} = \frac{1}{4}(r_1 + r_2 + c) \tag{3.13}$$

$$K = 1 - \frac{c}{2a_{min}} = \frac{r_1 + r_2 - c}{r_1 + r_2 + c} \tag{3.14}$$

$$\alpha = 2\arcsin\sqrt{\frac{a_{min}}{a}} \tag{3.15}$$

$$\beta = 2\arcsin\sqrt{K\frac{a_{min}}{a}} \tag{3.16}$$

The time of flight depends on which case. There are four total:

a) A includes no focal point

b) A includes only the non-body focal point

c) A includes only the body focal point

d) A includes both focal points

Then:

$$a)\quad t_f = \sqrt{\frac{a^3}{\mu}}\left[(\alpha - \sin\alpha) - (\beta - \sin\beta)\right] \tag{3.17}$$

$$b)\quad t_f = \sqrt{\frac{a^3}{\mu}}\left[2\pi - (\alpha - \sin\alpha) - (\beta - \sin\beta)\right] \tag{3.18}$$

$$c)\quad t_f = \sqrt{\frac{a^3}{\mu}}\left[(\alpha - \sin\alpha) + (\beta - \sin\beta)\right] \tag{3.19}$$

$$d)\quad t_f = \sqrt{\frac{a^3}{\mu}}\left[2\pi - (\alpha - \sin\alpha) + (\beta - \sin\beta)\right] \tag{3.20}$$

It is only valid for elliptical trajectories. For parabolic:

$$6\sqrt{\mu}t_f = (r_1 + r_2 + c)^{3/2} \mp (r_1 + r_2 - c^{3/2}) \tag{3.21}$$

And hyperbolic:

$$t_f = \sqrt{\frac{-a^3}{\mu}}\left[(\sinh\alpha' - \alpha') \mp (\sinh\beta' - \beta')\right] \tag{3.22}$$

L4 Transfer Orbits

4.1 A - Coplanar: Assumptions & Rocket Thrust

All equations are derived from Newton's laws. We assume the absence of external forces, resulting in rest or a constant velocity.

We work in a pseudo-inertial frame: a system for which Newtonian mechanics holds.

A quasi-inertial frame is a finite region where particles move without acceleration and according to special relativity. In such a frame, we correct for violations of laws of classical mechanics, and choose appropriate coordinate-time reference.

If a non-inertial reference frame is used, we use apparent forces: rectilinear acceleration of the origin of the frame, rotation of the reference frame (centrifugal or Coriolis), variable rate of rotation of the reference frame (Euler), all proportional to the mass of the body being acted upon.

4.1.1 Maneuvers with Rocket Thrust

Rocket propulsion comes from the expulsion of propellant. The relationship between change in mass and time is:

$$\frac{dM}{M} = -\frac{f}{V_j}dt \tag{4.1}$$

From this, we get:

$$\ln\frac{M_0}{M_e} = \int_0^{t_e}\frac{f}{V_j}dt \tag{4.2}$$

Note that the ratio is always greater than 1. For the maximum payload ratio $\frac{M_1}{M_0}$ we want a minimum mass ratio, and thus a minimum integral $\int f dt$, assuming a constant V_j. We get Tsiolkovsky's rocket equation:

$$\Delta V = V_j \ln\frac{M_0}{M_e} \tag{4.3}$$

We can construct the following EoM for a vehicle:

$$\frac{dV}{dt} = \frac{\dot{m}V_j}{M}\cos\left(\delta - \gamma\right) - g\sin\gamma \tag{4.4}$$

Usually, $\delta \approx \gamma$. Then:

$$\Delta V = V_j \ln\frac{M_0}{M_e} - \int_{t_0}^{t_e} g\sin\gamma dt \tag{4.5}$$

This can be written as:

$$\Delta V = \Delta V_{ideal} - \Delta V_G - \Delta V_D \tag{4.6}$$

Note that the third term is not assumed in the previous form as it is quite negligible.

Assuming an impulsive shot($\Delta t \to 0$):

$$\Delta V = V_j \ln\frac{M_0}{M_e} \tag{4.7}$$

We can see that the maximum payload is equivalent to the minimum ΔV.

The geometry also makes a difference. Generally, we have that:

$$\vec{V}_1 = \vec{V}_0 + \Delta\vec{V} \tag{4.8}$$

Looking at the change in angular momentum:

$$\Delta\vec{h} = \vec{r}_0 \times \vec{V}_1 - \vec{r}_0 \times \vec{V}_0 = \vec{r}_0 \times \Delta\vec{V} \tag{4.9}$$

From this, we see that we get a maximum change when r_0 is the greatest, and $\Delta\vec{V} \perp \vec{r}_0$. To preserve the direction of the angular momentum, you should aim the $\Delta\vec{V}$ in the direction of the orbital plane.

Looking at the change in energy:

$$\Delta\mathcal{E} = \frac{1}{2}\left(\left(\vec{V}_0 + \Delta\vec{V}\right)\cdot\left(\vec{V}_0 + \Delta\vec{V}\right) - V_0^2\right) = \frac{1}{2}\left(\Delta V\right)^2 + \vec{V}_0 \cdot \Delta\vec{V} \tag{4.10}$$

From this we can see that the maximum change in orbtal energy is obtained when ΔV is executed where V_0 is maximum, and when ΔV is directed along the velocity vector.

4.2 B - Coplanar: Optimum Transfers

Coplanar simply means $\Delta i = 0$.

For multiple rocket stages with impulsive shots:

$$\Delta V_{tot} = \sum_{i=1}^{n} (\Delta V)_i = V_j \sum_{i=1}^{n} \ln \frac{M_{0_i}}{M_{e_i}} \tag{4.11}$$

We assume that there is no rocket staging, and the same rocket engine is used. We get:

$$\Delta V_{tot} = V_j \ln \frac{M_0}{M_e} \tag{4.12}$$

ΔV_{tot} is minimum when the propellant consumption is minimum, and the payload mass is then maximum.

4.2.1 Optimum Transfer

We go from a circular orbit to another circular orbit ($e_1 = e_2 = 0$). We start with a V_{c_1}, apply a ΔV_1, later apply a ΔV_2, and end up at a ΔV_{c_2}. We can find with the law of cosines that:

$$\Delta V_1^2 = V_{c_1}^2 + V_1^2 - 2V_1 V_{c_1} \cos\gamma_1 \tag{4.13}$$

$$\Delta V_2^2 = V_{c_2}^2 + V_2^2 - 2V_2 V_{c_2} \cos\gamma_2 \tag{4.14}$$

For the optimum transfer, we also use the following relations:

$$V_1^2 = V_{c_1} \left(2 - \frac{r_1}{p}\left(1 - e^2\right)\right) \tag{4.15}$$

$$V_2^2 = V_{c_2} \left(2 - \frac{r_2}{p}\left(1 - e^2\right)\right) \tag{4.16}$$

$$V_1^2 = V_{c_1}^2 \left(2 - \frac{1-e^2}{z}\right) \tag{4.17}$$

$$V_2^2 = V_{c_1}^2 \left(\frac{2}{n} - \frac{1-e^2}{z}\right) \tag{4.18}$$

with $z = \frac{p}{r_1}$ and $n = \frac{r_2}{r_1}$. We also use:

$$\cos^2\gamma = \frac{h^2}{r^2V^2} = \frac{\mu p}{r^2V^2} = \frac{\frac{p^2}{r^2}}{\frac{2p}{r} - \frac{p}{a}} = \frac{\frac{p^2}{r^2}}{\frac{2p}{r} - (1-e^2)} \tag{4.19}$$

We can then derive the following relations:

$$\Delta V_1^2 = V_{c_1} \left(3 - 2\sqrt{z} - \frac{1-e^2}{z}\right) \tag{4.20}$$

$$\Delta V_2^2 = V_{c_1} \left(\frac{3 - 2\sqrt{\frac{z}{n}}}{n} - \frac{1-e^2}{z}\right) \tag{4.21}$$

Then:

$$\frac{\Delta V_{tot}}{V_{c_1}} = \frac{\Delta V_1 + \Delta V_2}{V_{c_1}} = \sqrt{3 - 2\sqrt{z} - \frac{1-e^2}{z}} + \sqrt{\frac{3 - 2\sqrt{\frac{z}{n}}}{n} - \frac{1-e^2}{z}} \tag{4.22}$$

We want to find where this is minimum. For any transfer trajectory:

$$a(1-e) \leq r_1 \tag{4.23}$$

$$a(1+e) \geq r_2 \tag{4.24}$$

We can then get regions of feasible transfer orbits. Substituting them into the equation for the total ratio, we can see that $\frac{\Delta V_{tot}}{V_{c_1}}$ decreases along both conditions. From that, we can find the optimum point as the point where the Hohmann transfer occurs. For a Hohmann transfer, $r_p = r_1$, and $r_a = r_2$. We can find:

$$\frac{\Delta V_1}{V_{c_1}} = \sqrt{\frac{2n}{n+1}} - 1 \tag{4.25}$$

$$\frac{\Delta V_2}{V_{c_1}} = \sqrt{\frac{1}{n}}\left(1 - \sqrt{\frac{2}{n+1}}\right) \tag{4.26}$$

$$\frac{\Delta V_{tot}}{V_{c_1}} = (n-1)\sqrt{\frac{2}{n(n+1)}} + \sqrt{\frac{1}{n}} - 1 \tag{4.27}$$

For large values of n we have that $\frac{\Delta V_1}{V_{c_1}} \to \sqrt{2}$. $\frac{\Delta V_2}{V_{c_2}}$ reaches a maximum of 0.19 at $n = 5.9$. This makes a maximum of $\frac{\Delta V_{tot}}{V_{c_1}}$ of 0.54 at $n = 15.6$.

For $n > 3.3$, $\Delta V_{tot} > \sqrt{2} - 1$, making it take more ΔV to circularize than to escape.

Hohmann transfers result in the lowest ΔV. The time required is found by:

$$t_{f_H} = \pi\sqrt{\frac{a^3}{\mu}} = \pi\sqrt{\frac{(r_1+r_2)^3}{8\mu}} = \frac{1}{4}T_{c_1}\sqrt{\frac{(n+1)^3}{2}} \tag{4.28}$$

For large n, Hohmann trajectories are quite slow.

4.3 C - Coplanar: Faster Transfers

Faster transfers can be done in many ways. We first look at the case where $r_{dep} = r_1 = r_p$. This means departure from the periapsis, similar as in a Hohmann transfer, but now we do not arrive at apoapsis, but before. This means:

$$r_1 = a(1-e) \tag{4.29}$$

$$r_2 = \frac{a(1-e^2)}{1+e\cos\theta_t} \tag{4.30}$$

For this case, we can develop the following expression:

$$\frac{\Delta V_{tot}}{V_{c_1}} = \sqrt{3 - 2\sqrt{\frac{n(1-\cos\theta_t)}{1-n\cos\theta_t}} - \frac{2-n(1+\cos\theta_t)}{1-n\cos\theta_t}} + \sqrt{\frac{3}{n} - \frac{2}{n}\sqrt{\frac{1-\cos\theta_t}{1-n\cos\theta_t}} - \frac{2-n(1+\cos\theta_t)}{1-n\cos\theta_t}} \tag{4.31}$$

The loss, equal to the increase in total ΔV required is:

$$L = \frac{\Delta V_{tot} - (\Delta V_{tot})_H}{(\Delta V_{tot})_H} \cdot 100\% \tag{4.32}$$

The flight time can be computed using:

$$t_f = \sqrt{\frac{a^3}{\mu}}(E_t - e\sin E_t) \tag{4.33}$$

After some substitutions, we end up at:

$$\frac{t_f}{T_{c_1}} = \frac{1}{2\pi}\sqrt{\left(\frac{1-n\cos\theta_t}{2-n(1+\cos\theta_t)}\right)^3}\left(E_t - \frac{n-1}{1-n\cos\theta_t}\sin E_t\right) \tag{4.34}$$

The gain, the reduction of flight time, can be found using:

$$G = -\frac{t_f - f_{f_H}}{t_{f_H}} \cdot 100\% \tag{4.35}$$

For small n, initially ΔV_{tot} increases slightly, with a major decrease in t_f. For large n, t_f decreases strongly. Looking at the loss and gain, the loss can be more than 100%. The gain is already over 40% at $\theta_t = 150°$.

4.4 D - Coplanar: Multiple Hohmann Transfers

Two cases are considered:

1. Transfer with a low apoapsis
2. Transfer with a high apoapsis

4.4.1 Transfer with a Low Apoapsis

Three impulsive shots are required:

$$\Delta V_1 = V_{c_1}\left[\sqrt{\frac{2r_a}{r_a+r_p}}-1\right] \tag{4.36}$$

$$\Delta V_2 = V_{c_1}\left[\sqrt{\frac{2r_2}{r_2+r_1}}-\sqrt{\frac{2r_a}{r_a+r_1}}\right] \tag{4.37}$$

$$\Delta V_3 = V_{c_1}\sqrt{\frac{r_1}{r_2}}\left[1-\sqrt{\frac{2r_1}{r_2+r_1}}\right] \tag{4.38}$$

Then:

$$\frac{\Delta V_{tot}}{V_{c_1}} = \sqrt{\frac{2r_2}{r_2+r_1}}+\sqrt{\frac{r_1}{r_2}}-\sqrt{\frac{r_1}{r_2}}\sqrt{\frac{2r_1}{r_2+r_1}}-1 \tag{4.39}$$

It can be seen that the intermediate apoapsis r_a does not affect the total ΔV at all. We can find the expression:

$$\frac{\Delta V_{tot}}{V_{c_1}} = (n-1)\sqrt{\frac{2}{n(n+1)}}+\sqrt{\frac{1}{n}}-1 \tag{4.40}$$

This is identical to the optimum transfer. Conducting multiple Hohmann transfers does thus not affect the total amount of ΔV required. This is true for $r_a < r_2$.

We can still need it, for:

1. Phasing of orbits
2. Multiple thrusting periods, reducing the gravity loss at departure and arrival at target

4.4.2 Transfer with a High Apoapsis

Now, $r_a > r_2$. Again, three impulsive shots are required:

$$\Delta V_1 = V_{c_1}\left[\sqrt{\frac{2m}{m+1}}-1\right] \tag{4.41}$$

$$\Delta V_3 = V_{c_1}\sqrt{\frac{1}{m}}\left[\sqrt{\frac{2n}{m+n}}-\sqrt{\frac{2}{m+1}}\right] \tag{4.42}$$

$$\Delta V_2 = V_{c_1}\sqrt{\frac{1}{n}}\left[\sqrt{\frac{2m}{m+n}}-1\right] \tag{4.43}$$

with $m = \frac{r_a}{r_1}$. We can find:

$$\frac{\Delta V_{tot}}{V_{c_1}} = (m-1)\left[\sqrt{\frac{2}{m(m+1)}}+\sqrt{\frac{2(n+m)}{nm}}-\sqrt{\frac{1}{n}}-1\right] \tag{4.44}$$

To find the conditions when this is more efficient, we define:

$$\Delta = \left(\frac{\Delta V_{tot}}{V_{c_1}}\right)-\left(\frac{\Delta V_{tot}}{V_{c_1}}\right)_2 < 0 \tag{4.45}$$

Plugging it in and inspecting it, we can consider:

1. $m = n$: $\Delta = 0$ - The three-impulse transfer is equal to the two-impulse.
2. $n = 1$, $m \to \infty$: $\Delta \to 2(\sqrt{2} - 1)$

We can find that for $n < 11.939$, the two-impulse is always optimal. For $11.939 < n < 15.582$, the three-impulse is optimal for large enough m. For $n > 15.582$, the three-impulse is always optimal.

Looking at the gain, we can see that for values of $m \leq 50$, the gain is 2% or less. For $m > 50$, the gain can be $4 - 6\%$, but the time of flight can become significant.

4.5 E - Coplanar: Circular & Elliptical Transfers

4.5.1 Circular to Elliptical

In the first case, we have that $e_1 = 0$ and $e_2 \neq 0$. There are two types of transfer:

I Joint the final ellipse at periapsis

II Join the final ellipse at apoapsis

For type I, we can find:

$$\Delta V_1 = V_{c_1}\sqrt{1+e_t} - V_{c_1} \tag{4.46}$$
$$\Delta V_2 = V_{2_p} - V_{c_{2_p}}\sqrt{1-e_t} \tag{4.47}$$

where $e_t = \frac{r_{2_p} - r_1}{r_{2_p} + r_1}$. We can also find:

$$V_{c_{2_p}} = \sqrt{\frac{\mu}{r_{2_p}}} = V_{c_1}\sqrt{\frac{r_1}{a(1-e)}} \tag{4.48}$$
$$V_{2_p} = \sqrt{\frac{\mu}{a}\left(\frac{1+e}{1-e}\right)} = V_{c_1}\sqrt{\frac{r_1}{a}\left(\frac{1+e}{1-e}\right)} \tag{4.49}$$

Then, we can find:

$$\Delta V_1 = V_{c_1}\left[\sqrt{\frac{2a(1-e)}{a(1-e)+r_1}} - 1\right] \tag{4.50}$$
$$\Delta V_2 = V_{c_1}\left[-\frac{r_1}{a}\sqrt{\frac{2}{(1-e)(1-e+\frac{r_1}{a})}} + \sqrt{\frac{r_1}{a}\left(\frac{1+e}{1-e}\right)}\right] \tag{4.51}$$
$$\left(\frac{\Delta V_{tot}}{V_{c_1}}\right)_I = \sqrt{\frac{2}{(1-e)(1-e+\frac{r_1}{a})}}\left(1 - e - \frac{r_1}{a}\right) + \sqrt{\frac{r_1}{a}\left(\frac{1+e}{1-e}\right)} - 1 \tag{4.52}$$

Similarly for type II, we can find:

$$\left(\frac{\Delta V_{tot}}{V_{c_1}}\right)_{II} = \sqrt{\frac{2}{(1+e)(1+e+\frac{r_1}{a})}}\left(1 + e - \frac{r_1}{a}\right) + \sqrt{\frac{r_1}{a}\left(\frac{1-e}{1+e}\right)} - 1 \tag{4.53}$$

Then, we can see that type II is always preferable. Injection at apoapsis requires less propellant.

Some observations:

1. For $0 < e < 0.6$, ΔV_{tot} is maximum for $16 < \frac{a}{r_1} < 27$
2. The loss of choosing type I instead of type II for $e < 0.4$ shows that type I requires up to 20% more ΔV
3. For $0.5 < e < 0.8$, type I can require up to 45% more ΔV

4.5.2 Elliptical to Circular

Again there are two types:

I Departure at periapsis

II Departure at apoapsis

We can find that type I is always preferable using a similar approach as before. Some observations:

1. ΔV_{tot} decreases when e increases for type I
2. The loss of choosing type II instead of type I shows for large e and $\frac{r_2}{a}$ a loss even up to $100 - 200\%$

4.5.3 Circular to Intersecting Elliptical

There are three types:

I Single impulse maneuver at intersection point

II A second impulse at apoapsis of final elliptical orbit

III A second impulse at periapsis of final elliptical orbit

From analysis, we can find that:

1. Type I requires the most ΔV_{tot}.
2. Type II always uses the least ΔV_{tot}
3. Type III for small e only uses slightly more than type II, and has a shorter flight time

4.6 F - Non-Coplanar

4.6.1 Geometry of Orbital Plane Changes

We need to look at the RAAN: the angle between the x axis and the point where the orbital plane of the spacecraft intersects the equatorial plane. We can then consider the inclination as the angle between the planes. We can then locate the spacecraft by looking at the argument of perigee, the angle between the node vector and the periapsis vector. The true anomaly then gives the angle between the periapsis and the position. Thus, the argument of latitude is given by:

$$u = \omega + \theta \tag{4.54}$$

Placing a spacecraft in an arbitrary new orbital plane, in general the following change:

1. inclination
2. RAAN

This creates an oblique spherical triangle, e.g. two arguments of latitude, and the angle between the two orbits. We apply spherical trigonometry to analyze this. Some assumptions:

1. The impulsive shot is conducted over the northern hemisphere ($u_1, u_2 < 180°$, $\Omega_2 > \Omega_1$, $i_2 > i_1$)
2. The oblique spherical triangle is defined by $i_1, \pi - i_2, A, u_1, u_2, \Omega_2 - \Omega_1$

In general, $A \neq \alpha$. This is because the orbits are projected on a celestial sphere, and orbits have a radial velocity component. However, we do assume $A = \alpha$. This is almost true for low e.

In general, to achieve a Δi or $\Delta\Omega$, other orbital elements are also affected. To transfer between two non-coplanar elliptical orbits, multiple impulsive shots are required. We cover three simple 3D transfer maneuvers.

4.6.2 Changing RAAN Without Inclination

This looks like the orbital plane rotating about the polar axis. Assuming a transfer between two circular orbits, the velocities are equal. Then:

$$(\Delta V)^2 = V_1^2 + V_2^2 - 2V_1V_2\cos\alpha \tag{4.55}$$

results in:

$$\frac{\Delta V}{V_1} = 2\sin i_1 \sin\frac{1}{2}\Delta\Omega \tag{4.56}$$

This tells us:

1. Given a V_1 and desired $\Delta\Omega$, the ΔV depends on the initial inclination
2. You should change Ω when V_1 is smaller
3. When the altitude is large, less ΔV is required for a $\Delta\Omega$

4.6.3 Changing the Inclination Without RAAN

This looks like the orbital plane rotating about the line of nodes. Again assuming two circular orbits, we can find that the maneuver should be executed at the ascending or descending node. We can find:

$$\cos A = \cos i_1 \cos i_2 + \sin i_1 \sin i_2 \cos\Delta\Omega \tag{4.57}$$

$$A = \Delta i \tag{4.58}$$

From the circular orbits:

$$\frac{\Delta V}{V_1} = 2\sin\frac{1}{2}\Delta i \tag{4.59}$$

From this:

1. Change inclination when V_1 is smaller
2. Large plane changes are costly
3. When the altitude is large, less ΔV is required for a Δi

The geometry of a Δi maneuver at the ascending node indicates that the angle β:

$$\beta = \frac{1}{2}(\pi + \Delta i) \tag{4.60}$$

Thus, the ΔV is not perpendicular to the initial orbital frame. This is because there is a component which opposes V_1 to keep $V_1 = V_2$. For small Δi, $\beta \approx 90°$.

4.6.4 Two Circular Orbits

A more general case is to transfer from a low altitude circular orbit to a higher circular orbit with a different inclination. The location of the first impulsive shot ΔV_1 becomes the periapsis of the transfer orbit. It is directed perpendicular to r_1. It then departs with inclination i_t. We then get:

$$\Delta V_1^2 = V_{c_1}^2 + V_{t_p}^2 - 2V_{c_1}V_{t_p}\cos(i_t - i_1) \tag{4.61}$$

Then, ΔV_2 is directed perpendicular to r_2 at apoapsis. From that:

$$\Delta V_2^2 = V_{c_2}^2 + V_{t_a}^2 - 2V_{c_2}V_{t_a}\cos(i_2 - i_t) \tag{4.62}$$

Then:

$$\frac{\Delta V_1}{V_{c_1}} = \sqrt{\frac{3n+1}{n+1} - 2\sqrt{\frac{2n}{n+1}}\cos(\Delta i^*)} \tag{4.63}$$

$$\frac{\Delta V_2}{V_{c_1}} = \sqrt{\frac{1}{n}\left[\frac{n+3}{n+1} - 2\sqrt{\frac{2}{n+1}}\cos(\Delta i - \Delta i^*)\right]} \tag{4.64}$$

with:

$$\Delta i^* = i_t - i_1 \tag{4.65}$$
$$n = \frac{r_2}{r_1} \tag{4.66}$$

There is an optimum value of Δi^* where ΔV_{tot} is minimum. We can see:

1. $\Delta i^* << \Delta i$ (most inclination change should be at the apoapsis)
2. Δi^*_{max} occurs at $n < 3$

We can consider the total ΔV when using the optimal i_t. We can see:

1. For $\Delta i \approx 30°$, $\Delta V_{tot} \approx \frac{1}{2}V_{c_1}$
2. For $n > 5$, $\Delta V_{tot} \approx 0.5 - 0.6V_{c_1}$

Looking at the loss:

1. Greatest for small inclination changes
2. For KSC to GEO, loss is about 0.6%

Looking at the geometry, we want to find the thrust angles. Expressions are on slides. We can see:

1. For $n \downarrow 1$, thrust is closer to perpendicular to the orbital plane
2. As n increases, β decreases
3. For a given n, β increases with Δi

4.6.5 Inclination Change at High Altitude

We know that a large inclination change is more effective at a high altitude.

We can do a following transfer:

1. Hohmann transfer to high apoapsis r_a at i_1
2. Full Δi at apoapis
3. Hohmann transfer down to r_2 at i_2

Looking at the total ΔV, we can observe:

1. The total decreases monotonically for increasing n
2. For $n \approx 1$, $\beta_2 = 95 - 115°$
3. β_2 decreases monotonically for increasing n

Looking at the gain:

1. For small Δi, the gain is small, and only for large m
2. For large Δi, the gain could be up to 5% for small n and decently sized m

L5 Lunar Trajectories

5.1 A - Earth-Moon System

Look at reader.

5.2 B - First Steps

We perform it as a simple first-order analysis as a two-body problem, neglecting the Moon's gravitational force, and a trajectory which is a conic section. Also assuming a circular coplanar lunar orbit, we can approximate optimum launch dates, injection conditions, and flight times.

Given r_1, V_1, and γ_1, we want to find θ_1 and θ_2. Calculating the eccentric anomalies:

$$E = 2\arctan\left[\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right] \tag{5.1}$$

The time of flight:

$$t_f = \sqrt{\frac{a^3}{\mu}}\left[E_2 - E_1 - e\left(\sin E_2 - \sin E_1\right)\right] \tag{5.2}$$

Observations:

1. Small increase in V_1 leads to a large reduction in Δt_f
2. Phasing orbits have been used

5.3 C - Next Steps

5.3.1 2D with Patched Conics

We limit ourselves to coplanar orbits. We account for the lunar gravitation with a series of patched conics.

1. A trajectory in the SOI of the Earth where only the Earth's gravity is considered to be acting
2. A trajectory within the SOI of the Moon, in which only the Moon's gravity is considered to be acting

This approximation is sufficiently good for mission analysis.

The geocentric phase is specified by r_1, V_1, γ_1, and ψ_1 (lunar phase angle at departure). More convenient to select are r_1, ΔV_0, δ_0 (thrust angle), and λ_2 (angle where Moon's SOI is entered).

We assume $\theta_2 < 180°$. Then, at departure:

$$\Delta V_0^2 = V_0^2 + V_1^2 - 2V_0V_1\cos\gamma_1 \tag{5.3}$$

Using the other variables:

$$V_1^2 = V_0^2 + \Delta V_0^2 + 2V_0\Delta V_0\cos\delta_0 \tag{5.4}$$

At arrival at the Moon's SOI:

$$r_2 = \sqrt{r_M^2 + R_{SOI}^2 - 2r_M R_{SOI}\cos\lambda_2} \tag{5.5}$$

The arrival conditions determine the initial conditions of the selenocentric trajectory. To patch, we need relations between the frames:

$$\vec{r}_{a/c} = \vec{r}_{a/b} + \vec{r}_{b/c} \tag{5.6}$$

$$\rightarrow \vec{V}_3 = \vec{V}_2 - \vec{V}_M \tag{5.7}$$

We can find:

$$V_3 = \sqrt{V_2^2 + V_M^2 - 2V_2V_M \cos(\gamma_2 - \beta_2)} \tag{5.8}$$

$$V_3 \sin \epsilon_3 = V_M \cos \lambda_2 - V_2 \cos(\lambda_2 - \gamma_2 + \beta_2) \tag{5.9}$$

$$\epsilon_3 = \arcsin \left[\frac{V_M}{V_3} \cos \lambda_2 - \frac{V_2}{V_3} \cos(\lambda_2 - \gamma_2 + \beta_2) \right] \tag{5.10}$$

Given selenocentric initial conditions, the hyperbolic portion of the trajectory can be computed including perilune. There are three classes of lunar missions:

1. Lunar impact ($r_p < R_M$)
2. Lunar orbit
3. Circumlunar flight

L6 (1) Interplanetary Trajectories

6.1 A - Patched Conics

We already looked at the method of patched conics for Lunar trajectories. We did:

1. A trajectory using only Earth's gravity
2. A trajectory with only the Moon's gravity
3. Patching the two trajectories together

For interplanetary:

1. Escape the gravitational field of the departure planet
2. Patch the planetocentric to the heliocentric trajectory
3. Follow a transfer trajectory from departure to target planet
4. Patch the heliocentric to the planetocentric trajectory
5. Enter the gravitational field of the target planet
6. Impact, capture, or fly-by

Assumptions:

1. Planetary orbits are assumed to be circular and coplanar

6.1.1 Heliocentric Trajectory

Using the energy equation, we can get:

$$V_{\oplus/\odot} = \sqrt{\frac{\mu_\odot}{r_\oplus}} \tag{6.1}$$

$$V_{♂/\odot} = \sqrt{\frac{\mu_\odot}{r_♂}} \tag{6.2}$$

For the transfer trajectory:

$$-\frac{\mu_\odot}{r_\oplus + r_♂} = \frac{V_\diamond^2}{2} - \frac{\mu_\odot}{r_\diamond} \tag{6.3}$$

From this:

$$V_{\diamond/\odot|\text{dep}} = \sqrt{\frac{2\mu_\odot}{r_\oplus}}\sqrt{\frac{r_♂}{r_\oplus + r_♂}} \tag{6.4}$$

$$V_{\infty|\text{dep}} = \left|\sqrt{\frac{2\mu_\odot}{r_\oplus + r_♂}} - \sqrt{\frac{\mu_\odot}{r_\oplus}}\right| \tag{6.5}$$

And equations for target.

6.1.2 Sphere of Influence

$$r_{SOI} = a_i \left(\frac{m_i}{m_\odot}\right)^{\frac{2}{5}} \tag{6.6}$$

For a hyperbolic escape trajectory, $r_{SOI} \approx r_\infty$.

6.1.3 Exiting the SOI

Patching is based on relative motion. We get:

$$\vec{r}_{\diamond/\odot} = \vec{r}_{\diamond/\oplus} + \vec{r}_{\oplus/\odot} \tag{6.7}$$

And we can take derivatives with respect to time to get velocities. The second term would be V_∞.

6.1.4 Departure Trajectory

From the energy equation applied to the hyperbolic trajectory:

$$V_{p_{\diamond/\oplus}} = \sqrt{V^2_{\infty_{\diamond/\oplus}} + \frac{2\mu_\oplus}{r_{p_{\diamond/\oplus}}}} \tag{6.8}$$

From this, ΔV_{dep} can be found.

6.1.5 Target Trajectory

Energy equation:

$$V_{p_{\diamond/♂}} = \sqrt{V_{\infty/♂} + \frac{2\mu_{♂}}{r_{p_{\diamond/♂}}}} \tag{6.9}$$

From this, ΔV_{tar} can be found.

Then, we can get ΔV_{tot}.

6.1.6 Departure and Target Timing

Based on rotational kinematics:

$$\theta_{i_2} = \theta_{i_1} + \omega_i(t_2 - t_1) \tag{6.10}$$

Doing this for the target and departure planets, we can make an expression for the phase angle, indicating when we need to set out on the heliocentric trajectory.

Given the relationship, we can get the synodic period as:

$$T_{synodic} = \frac{2\pi}{|\omega_1 - \omega_2|} \tag{6.11}$$

6.2 B - General Aspects and Assumptions

Until the 1970's, direct transfer trajectories were used. These required high ΔV. We can also use gravity assist maneuvers. This exchanges momentum with another planet. The velocity vector is changed in direction in the planetocentric, and also magnitude in heliocentric. This does result in longer trajectories and flight times.

Same assumptions and approximations as before are used.

6.3 C - Optimum 2D Trajectories

We look at feasible trajectories from the Earth to Mars. Looking at the velocity, we can get using the law of cosines:

$$\frac{\Delta V_0}{V_e} = \sqrt{2c_1 + 3 - 2\sqrt{z} - \frac{1-e^2}{z}} - \sqrt{c_1} \tag{6.12}$$

$$\text{with } c_1 = \left(\frac{V_{c_0}}{V_e}\right)^2 \tag{6.13}$$

From this, we can find infeasible trajectories. We again see that the most efficient trajectory is the Hohmann trajectory. We can calculate the excess velocities at departure and arrival, and the ΔV for both for the Hohmann transfer. We can see:

1. Minimum required ΔV_0 is to Venus
2. Minimum required ΔV_3 is at Mars
3. For Jupiter and Saturn it is high
4. Escape from solar system at Earth is quite close to Pluto and Neptune
5. Less propellant is required to escape the solar system than to transfer via Hohmann trajectory and entering a circular orbit (except for Mars and Venus)

6.4 D - Orbital Insertion

To insert into an orbit about a target planet, we need to consider:

1. Parking orbit on departure (near circular, low altitude)
    - Near circular
    - Low altitude
    - Maximum change in energy for highest velocity
2. Orbit on arrival at the target
    - Low propellant consumption implies highly eccentric orbit
    - ΔV applied at periapsis and $\perp$ to r_p
    - There may be mid-course corrections (DSMs)

Assessing the limits of the parameters for insertion:

1. $\Delta V_3 > \sqrt{\frac{2\mu}{r_r} + V_{\infty_t}^2} - \sqrt{\frac{2\mu}{r_3}}$ as we want to enter a closed orbit
2. As e_t decreases, ΔV_3 increases
3. For a circular orbit, $\Delta V_3 = \sqrt{\frac{2\mu}{r_r} + V_{\infty_t}^2} - \sqrt{\frac{\mu}{r_3}}$

6.4.1 Insertion into Orbits with a Specified Eccentricity

We can find:

$$\Delta V_3 = \sqrt{\frac{2\mu}{r_3} + V_{\infty_t}^2} - \sqrt{\frac{\mu}{r_3}(1+e)} \tag{6.14}$$

We can see that ΔV_3 decreases as e increases.

Given an e and V_{∞_t}, we now want to find the optimum r_3 for which ΔV_3 is minimum. For this, we take the derivative, we can find:

$$\frac{r_3}{R}|_{opt} = \left(\frac{V_{esc_{surf}}}{V_{\infty_t}}\right)^2 \frac{1-e}{1+e} \tag{6.15}$$

$$\text{with } V_{esc_{surf}} = \sqrt{\frac{2\mu}{R}} \tag{6.16}$$

Considering the physical limits, we need $\frac{r_3}{R}|_{opt} \geq 1$. From this, we can get a limit e. For $e < e_{lim}$, the ratio decreases for increasing e and V_{∞_t}. For $e \geq e_{lim}$, the ratio is always 1. The absolute minimum ΔV_3 is at $e = 1$. Then:

$$\Delta V_3|_{\text{min, abs}} = V_{esc_{surf}} \left[\sqrt{1 + \left(\frac{V_{\infty_t}}{V_{esc_{surf}}}\right)} - 1\right] \tag{6.17}$$

6.4.2 Insertion into Orbits with a Specified Semi-Major Axis

a determines the period, a key parameter for missions. Note that:

1. For a given e, if r_3 increases, r_a increases
2. For a given a, if r_3 increases, r_a decreases

We can get:

$$\Delta V_3 = \sqrt{\frac{2\mu}{r_e} + V_{\infty_t}^2} - \sqrt{\frac{2\mu}{r_3} - \frac{\mu}{a}} \tag{6.18}$$

We can again take a partial derivative with respect to r_3. Since $a > 0$, this derivative is also greater than 0 for all r_3. As r_3 increases, ΔV_3 increases. Thus, ΔV_3 is minimum for minimum r_3. Assuming $\frac{r_3}{R}|_{opt} = 1$:

$$\Delta V_3|_{min} = V_{esc_{surf}} \left[\sqrt{1 + \left(\frac{V_{\infty_t}}{V_{esc_{surf}}} \right)^2} - \sqrt{1 - \frac{R}{2a}} \right] \tag{6.19}$$

Thus, $\Delta V_3|_{min}$ increases for increasing V_{∞_t} or decreasing a. For a parabola, we get the same equation as before.

6.5 E - Approach and Flyby

Some notable parameters:

1. Impact parameter B
2. Aiming point
3. Arrival velocity
4. β angle
5. Asymptotic deflection angle or turning angle α

Using constant angular momentum:

$$BV_{\infty_t} = r_3 V_3 \tag{6.20}$$

Using vis-viva:

$$V_3^2 = V_{3,esc}^2 + V_{\infty_t}^2 = \frac{2\mu}{r_3} + V_{\infty_t}^2 \tag{6.21}$$

Combining, we can get equations for r_3 and V_3. Then, we can find:

$$\sin \frac{1}{2}\alpha = \frac{1}{\sqrt{1 + \frac{B^2 V_{\infty_t}^4}{\mu^2}}} \tag{6.22}$$

With the impact condition $r_3 < R$, an expression can be made for $\frac{B}{R}$.

We can define a capture radius, where if the asymptote intersects a sphere with that radius, impact will occur. To use atmospheric braking, the aiming point lies within a small range forming an annulus-shaped (re-)entry corridor.

$\alpha_{\max}$ is the maximum angle through which the velocity vector can be rotated, which is a direct indicator for the boost in heliocentric orbital energy.

Given a V_{∞_t} and a desired r_3, the aiming distance B can be determined.

6.5.1 F - Gravity-Assist Maneuvers

Relative motion relates planetocentric and heliocentric velocities. We have:

$$\vec{V}_2 = \vec{V}_t + \vec{V}_{\infty_t} \tag{6.23}$$

$$\vec{V}_4 = \vec{V}_t + \vec{V}_{\infty_t}^{\star} \tag{6.24}$$

From this:

$$\Delta\vec{V} = \vec{V}_4 - \vec{V}_2 = \vec{V}^{\star}_{\infty_t} - \vec{V}_{\infty_t} \tag{6.25}$$

Using the geometry:

$$\Delta V = 2V_{\infty_t} \sin\frac{1}{2}\alpha \tag{6.26}$$

Considering the change in total specific energy, change in potential energy is negligible, but the kinetic not. We can eventually get:

$$\Delta\mathcal{E} = 2V_t' V_{\infty_t} \sin\frac{1}{2}\alpha \cos\beta = V_t' \Delta V \cos\beta \tag{6.27}$$

We can then write the key performance parameters in terms of the design parameters B, V_{∞_t} and β. Some observations:

1. ΔV is independent of β
2. Given V_{∞_t} and β, as $B\downarrow$, $\Delta V, \delta\mathcal{E}, \Delta a^{-1}\uparrow$
3. For an increase in energy, the spacecraft must pass behind the planet
4. Given $B, V_{\infty_T}, \beta, \mu$, the maximum change in orbital energy is when V_t' is maximum

L6 (2) Three-Body Problem

6.1 Equations of Motion

In general:

1. Three bodies
2. All free to move in 3D space
3. Bodies act only under their mutual gravitation

The inertial reference frame has xyz axes and the origin at an arbitrary location O. We get with respect to the origin:

$$\frac{d^2 r_i}{dt^2} = G\frac{m_j}{r_{ij}^3} r_{ij} + G\frac{m_k}{r_{ik}^3} r_{ik} \tag{6.1}$$

We thus have three second order vector diffrential equations. No general closed-form solutions have been found. There are partial solutions for special cases, and solutions involving series expansions.

We can put the equation in Lagrange form, where the vectors are not with respect to the origin. We can get:

$$\frac{d^2 \vec{r}_{ij}}{dt^2} = G\left[m_j\left(\frac{\vec{r}_{ij}}{r_{ij}^3} + \frac{\vec{r}_{jk}}{r_{jk}^3}\right) - (m_k + m_i)\frac{\vec{r}_{ki}}{r_{ki}^3}\right] \tag{6.2}$$

The Jacobi form is the most powerful for insight. It is non-symmetric, and based on two key vectors, r_{12} and R. It forms the basis for lunar theory and triple stellar system problems.

The $\vec{R}$ is pointed from $\vec{r}_{12}$'s origin (barycenter of 1 and 2) to 3. We get two second order differential equations:

$$\frac{d^2 \vec{r}_{12}}{dt^2} = -G\left[(m_1 + m_2)\frac{\vec{r}_{12}}{r_{12}^3} + m_3\left(\frac{\vec{r}_{13}}{r_{13}^3} - \frac{\vec{r}_{23}}{r_{23}^3}\right)\right] \tag{6.3}$$

$$\frac{d^2 \vec{R}}{dt^2} = -GM\left[\alpha\frac{\vec{r}_{13}}{r_{13}^3} + (1-\alpha)\frac{\vec{r}_{23}}{r_{23}^3}\right] \tag{6.4}$$

$$\text{with } M = m_1 + m_2 + m_3 \tag{6.5}$$

$$\text{and } \alpha = \frac{m_1}{m_1 + m_2} \tag{6.6}$$

Making some simplifications, like $\alpha \approx 1$ and $r_{13} \approx R$, we can get:

$$\frac{d^2 \vec{r}_{12}}{dt^2} = -G(m_1 + m_2)\frac{\vec{r}_{12}}{r_{12}^3} \tag{6.7}$$

$$\frac{d^2 \vec{R}}{dt^2} = -GM\frac{\vec{R}}{R^3} \tag{6.8}$$

These represent the superposition of two two-body trajectories, which are conic sections.

6.2 Circular Restricted

We assume:

1. $m_3 << m_1, m_2$
2. m_1 and m_2 move in circular orbits about the barycenter of the system

Consequences:

1. System goes from order 18 to order 6
2. m_1 and m_2 move in a single plane
3. Conservation of energy and angular momentum hold for 1 and 2 but not 3

We define the following:

$$r_1^2 = (\xi - \xi_1)^2 + (\eta - \eta_1)^2 + \zeta^2 \tag{6.9}$$
$$r_2^2 = (\xi - \xi_2)^2 + (\eta - \eta_2)^2 + \zeta^2 \tag{6.10}$$
$$\frac{d^2\vec{r}}{dt^2} = -G\frac{m_1}{r_1^3}\vec{r}_1 - G\frac{m_2}{r_2^3}\vec{r}_2 \tag{6.11}$$

Motion in inertial frame:

$$\frac{dr}{dt} = \frac{\delta r}{\delta t} + \omega \times r \tag{6.12}$$

above are vecotrs

Taking the derivative with respect to time, we can get an equation for the acceleration with respect to the rotating frame. We can identify:

1. Acceleration due to m_1 and m_2
2. Coriolis acceleration
3. Centrifugal acceleration

Using nondimensionalization using:

$$\mu = \frac{m_2}{m_1 + m_2} \tag{6.13}$$

We always choose $m_1 > m_2$, such that $\mu < 1/2$. Also:

$$OP_1 = \mu \tag{6.14}$$
$$OP_2 = 1 - \mu \tag{6.15}$$

Now, we choose $1/\omega$ as the unit of time:

$$t* = t\omega \tag{6.16}$$

We can rewrite the EoM using these nondimensionalizations. From now on, all variables are nondimensional.

Next, we write the vectors in terms of scalar Cartesian coordinates.

If we introduce the scalar potential function:

$$U = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \tag{6.17}$$

And carry out partial differentials, we can get EoM in terms of the scalar potential function.

We see U:

1. Accounts for gravitational and centrifugal
2. Produces a non-central force field
3. Is not time-dependent (thus conservative)

6.3 I - Jacobi C and Surfaces of Hill

We start with the CR3BP.

We multiply all EoM by $\dot{x}, \dot{y}, \dot{z}$ respectively, and adding them. Then, substituting the derivative of U, and integrating, we get:

$$V^2 = 2U - C \tag{6.18}$$

This is Jacobi's integral. C is Jacobi's constant, and V is the velocity with respect to the rotating frame.

If we solve for C, and substitute the potential function, we can find a new equation. This equation describes the surfaces of Hill.

The surfaces of Hill are those surfaces on which $V = 0$. They are:

1. symmetric wrt the xz and xy planes
2. symmetric wrt the yz plane when $\mu = 1/2$
3. contained within a cylinder around z axis of radius $C^{1/2}$
4. some asymptotic as $z^2 \to \infty$

Varying the value of C (a certain energy level), we change the positions particles are able to move to.

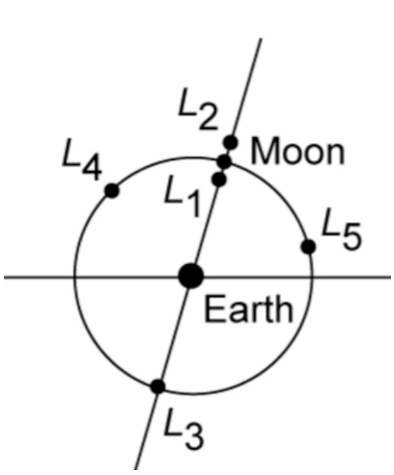
6.4 J - Lagrange Libration Points

Lagrange points are the points of equilibrium in which the acceleration on the particle m_3 is zero:

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} = \frac{\partial U}{\partial z} = 0 \tag{6.19}$$

First, we can find thus that $z = 0$, and thus points are in the xy plane. From the other 2 equations, we can find five points. We can find:

1. L_1, L_2, L_3 are found on the x axis
2. L_4, L_5 are located in the xy plane



L7 Relative Motion, Phasing, Rendezvous

7.1 A - Clohessy-Wiltshire Equations

Starting again with the CR3BP.

Differences:

1. P_1 is now P_0, and is at the origin of the new system
2. P_2 is now P_1, and is the satellite used as basis of motion
3. P_3 is now P_2, the second satellite
4. $\mu = 0$ as P_1 is massless
5. P_1 is in a circular orbit
6. P_1 is fixed wrt xyz

Then:

1. x' is radial
2. y' is along track
3. z' is cross track

Now:

$$x = 1 + x' \tag{7.1}$$
$$y = y' \tag{7.2}$$
$$z = z' \tag{7.3}$$

From the CR3BP and some linearizing ($r^{-3} \approx 1 - 3x'$):

$$\ddot{x}' - 2\dot{y}' = 3x' \tag{7.4}$$
$$\ddot{y}' + 2\dot{x} = 0 \tag{7.5}$$
$$\ddot{z}' = -z' \tag{7.6}$$

If we put it back in physical units, and add a constant force, we can easily identify the Coriolis and centrifugal accelerations.

$$\ddot{x}' - 2n_1\dot{y}' - 3n_1^2 x' = f_{x'} \tag{7.7}$$
$$\ddot{y}' - 2n_1\dot{x}' = f_{y'} \tag{7.8}$$
$$\ddot{z}' + n_1^2 z' = f_{z'} \tag{7.9}$$

7.2 B - Clohessy-Wiltshire Solution

We now drop the index ' and subscript 1.

We look for an analytical solution to the relative differential equations of motion. Starting with the equation for z, and looking at the homogeneous part, we have a simple harmonic oscillator. Also finding a particuklar solution, we can find:

$$z = A_1 \sin nt + B_1 \cos nt + \frac{f_z}{n^2} \tag{7.10}$$

We can do the same for x and y (equations are long icba to put them).

We see:

1. The period of the periodic terms is equal to the orbital period of satellite 1
2. If all forces are 0, the motions in the radial and cross track directions are purely harmonic, but in the along track there is also a secular component varying linearly with time (this is called drift), the oscillation in the along track direction is a quarter period ahead of the radial direction oscillation, the amplitude of the along track oscillation is twice the amplitude of the radial oscillation
3. When an additional force is applied to satellite 2, the x component yields a constant contribution in the radial direction and linear drift in the along track direction, the y component a linear drift in the radial direction and a constant contribution and a quadratic drift in the along track direction, and the z component a constant contribution in the cross track direction

7.3 C - Characteristics of the Motion

7.3.1 Unperturbed Relative Motion

Motion in the cross track direction is a pure harmonic oscillation, uncoupled from the other motions.

We manipulate the x and y equations. We have four constants showing up from the initial conditions. Then, squaring both equations and adding them, we can identify the shape of an ellipse. The motion thus represents an ellipse, with its center moving along the y axis with constant velocity.

$$b = \frac{1}{2}a \tag{7.11}$$

$$V_{y_c} = -3(2nx_0 + \dot{y}_0) \tag{7.12}$$

We can develop a whole range of specific motions with specific initial conditions.

7.3.2 Relative Motion after an Impulsive Shot

An impulsive shot defines the initial conditions after the ΔV. We can once again look at multiple versions of initial conditions to look at specific dynamics.

7.4 D - Phasing Orbits

We often need to design such that we arrive at a specific position at a specific time. A direct transfer is often not practical, and thus we prefer phasing orbits.

Both the spacecraft and the target position move with respect to inertial space. We use the terms the chaser and the target, or satellite 1 and satellite 2. In practice, we aim slightly behind the target, in a slightly lower orbit.

From coplanar transfers, we know that the number of intermediate orbits does not affect the energy requirement. For non-coplanar transfers, there is only a weak dependence on the inclination of the intermediate transfer orbit, and there is always an optimum.

7.4.1 Coplanar Direct Transfer

We start with a phase angle ψ_0 at t_0. From that, we start a direct transfer orbit at t_1 after a ϕ_1. At the end of the direct transfer, we arrive at satellite 2 at t_2 and ψ_2. We can find that:

$$t_f = t_2 - t_0 = \frac{\phi_1}{n_1} + t_t \tag{7.13}$$

Using angular relationships, we can find:

$$\psi_2 = \psi_0 - \left(1 - \sqrt{\left(\frac{r_1}{r_2}\right)^3}\right)\phi_1 + \pi\left(\sqrt{\left[\frac{1}{2}\left(1 + \frac{r_1}{r_2}\right)\right]^3} + 1\right) \tag{7.14}$$

For a rendezvous, $\psi = n2\pi$. We can then find ϕ_1, the wait time.

For $r_2 >> r_1$, we can get that $\psi_0 - \phi_1 = 116°$.

For $\Delta r = r_2 - r_1 << r_1$, we can linearize the equation, and find:

$$\phi_1 \approx \frac{2}{3}\frac{r_1}{\Delta r}\psi_0 - \frac{1}{2}\pi \tag{7.15}$$

From this, we can find that for ψ_0, $\phi_1 = 90°$. Thus in general, injection has to take place when satellite 1 is in the range of $90 - 160°$ behind satellite 2.

7.4.2 Non-Coplanar Direct Transfer

The phase angle now starts from the line of nodes instead of the location at t_0.

Injection occurs at $s_{1,1}$.

Now also not that injection can happen k revolutions later. Then:

$$(\Delta t_f)_k = 2\pi \frac{k}{n_1} \tag{7.16}$$

$$\Delta\psi_2 = 2\pi k \frac{n_2}{n_1} \tag{7.17}$$

7.4.3 Phasing Orbits

The phase angle interval is the basic interval α:

$$\alpha = 2\pi\sqrt{\left(\frac{r_1}{r_2}\right)^3} \tag{7.18}$$

If the square root is an integer, only the same discrete points in the target orbit can be achieved. Otherwise, any point can be reached, but it might take a very long time. For those cases, a phasing orbit is needed. We can perform a low or a high phasing orbit. For low:

1. Two ΔV at periapsis
2. One ΔV at apoapsis

For high:

1. One ΔV at periapsis
2. Two ΔV at apoapsis

7.5 E - Phases of Rendezvous Flight

A rendezvous flight is the precursor for a docking procedure. If the distance between the chaser and the target is not large, the relative motion can be described by the CW equations. We assume the chaser is in a lower orbit and in the same orbital plane.

We look at both the inertial and co-rotating reference frames.

The phases of a rendezvous flight:

1. Launch
2. Mid-course
3. Terminal
4. Docking

The terminal phase starts when there is a specific slant range between the chaser and the target.

7.5.1 Launch Phase

Options:

1. Launch directly into a coplanar orbit, behind and slightly below the target orbit
   - Long waiting times or impossibly tight constraints on launch time and injection accuracy
2. Launch into a much lower parking orbit
   - Faster phasing

7.5.2 Mid-Course Phase

Goal is to maneuver the chaser close enough to the target to enable guidance sensors to acquire the target.

The result is a constant drift rate of the chaser towards the target.

A hold point is a position in the rendezvous trajectory where the relative velocity is temporarily set to zero, allowing for checks, waits, etc.

7.5.3 Terminal Phase

The chaser is brought into an interception course with the target, which must be visible to the crew or sensors before the terminal phase initiation point.

The not preferred method uses two impulsive shots (which can have errors). Instead, guidance is according to forced rectilinear motion, or rendezvous along the line of sight.

7.6 F - Terminal Flight to Last Hold Point

We have initial conditions at t_0. Then, a ΔV is applied to initiate a trajectory which will arrive directly at the target. Then, another ΔV is applied to nullify the relative velocity at t_2, the last hold point. The CW equations are used to find the ΔV. Note that these are meant as a qualitative impression.

7.7 G - Terminal Flight from Last Hold Point

We have two cases:

1. Flight after a rocket pulse in x direction
2. Flight after a rocket pulse in y direction

7.7.1 Rocket Pulse in Radial Direction

We have two sub-cases:

1. Direct to target
2. Directly below the target

For the first, we have:

$$\Delta V_{tot} = \frac{1}{2} n y_0 \tag{7.19}$$

$$\frac{M_p}{M_0} = 1 - \exp\left(-\frac{n y_0}{2V_j}\right) \tag{7.20}$$

For the second:

$$\Delta V_{tot} = \frac{3}{2} n y_0 \tag{7.21}$$

$$\frac{M_p}{M_0} = 1 - \exp\left(-\frac{3 n y_0}{2 V_j}\right) \tag{7.22}$$

7.7.2 Rocket Pulse in Tangential Direction

We get:

$$\Delta V_{tot} = \frac{n y_0}{3 N \pi} \tag{7.23}$$

$$\frac{M_p}{M_0} = 1 - \exp\left(-\frac{n y_0}{3 N \pi V_j}\right) \tag{7.24}$$

more but im prolly not reading this later anyways.